Examen 2012

I- Bruit blanc - bruit coloré

A- Bruit blanc

$$\frac{dv}{dt} = -\gamma v + \Gamma(t)$$

1) $\langle \Gamma(t)\Gamma(t')\rangle = 2D\,\delta(t-t') = 2D \int e^{i\omega(t-t')} \underbrace{1_\omega}_{\hookrightarrow \text{uniforme.}} d\omega$ 1

2) $\frac{dv(t)}{dt} = -\gamma v \Rightarrow v(t) = K e^{-\gamma t}$

et $v'(t) = K' e^{-\gamma t} - \gamma K e^{-\gamma t} = -\gamma K e^{-\gamma t} + \Gamma(t)$

$\Rightarrow K' e^{-\gamma t} = \Gamma(t) \Rightarrow K'(t) = \Gamma(t) e^{\gamma t}$

$\Rightarrow K(t) = \int_0^t \Gamma(t') e^{\gamma t'} dt'$

et $v(t) = \int_0^t \Gamma(t') e^{\gamma(t'-t)} dt'$

<u>Au final:</u> $v(t) = K e^{-\gamma t} + \int_0^t \Gamma(t') e^{\gamma(t'-t)} dt'$ 1

ou

et $v(t=0) = K = v_0$

$\hookrightarrow \boxed{v(t) = v_0 e^{-\gamma t} + \int_0^t \Gamma(t') e^{\gamma(t'-t)} dt'}$ 2

$$v(t_1)\,v(t_2) = v_0^2\, e^{-\gamma(t_1+t_2)} + v_0\, e^{-\gamma t_1} \int_0^{t_2} \Gamma(t'_2)\, e^{-\gamma(t_2-t'_2)}\, dt'_2 \quad 2$$

$$+ v_0\, e^{-\gamma t_2} \int_0^{t_1} \Gamma(t'_1)\, e^{-\gamma(t_1-t'_1)}\, dt'_1$$

$$+ \int_0^{t_1} \Gamma(t'_1)\, e^{-\gamma(t_1-t'_1)}\, dt' \int_0^{t_2} \Gamma(t'_2)\, e^{-\gamma(t_2-t'_2)}\, dt'_2$$

$$\Rightarrow \langle v(t_1)\, v(t_2)\rangle = v_0^2\, e^{-\gamma(t_1+t_2)} + \int_0^{t_1}\int_0^{t_2} e^{-\gamma(t_1-t'_1+t_2-t'_2)} \langle \Gamma(t'_1)\Gamma(t'_2)\rangle$$

$$= v_0^2\, e^{-\gamma(t_1+t_2)} + 2D \int_0^{\min(t_1,t_2)} e^{-\gamma(t_1+t_2-2t'_1)}\, dt'_1 .$$

$$= v_0^2\, e^{-\gamma(t_1+t_2)} + e^{-\gamma(t_1+t_2)}\, 2D \int_0^{\min(t_1,t_2)} e^{2\gamma t'_1}\, dt'_1$$

$$= v_0^2\, e^{-\gamma(t_1+t_2)} + 2D\, \frac{1}{2\gamma}\left[e^{-\gamma|t_1-t_2|} - e^{-\gamma(t_1+t_2)}\right].$$

Dans la limite des grands temps on a :

$$\langle v(t_1)\, v(t_2)\rangle \simeq \frac{D}{\gamma}\, e^{-\gamma|t_1 - t_2|} \quad 2$$

$$v_0^2 e^{2\gamma t} + \frac{D}{\gamma} - v_0^2 e^{-2\gamma t}$$

4) $\sigma_v^2 = \langle v^2(t)\rangle - \langle v\rangle^2 = \frac{D}{\gamma}\left(1 - e^{-2\gamma t}\right)$ 1

$t = 0 \qquad \sigma_v^2 = 0$ car v fixée à v_0

$t \ll \gamma^{-1} \qquad \sigma_v^2 = 2Dt$ diffusion dans l'espace des vitesses

$t \gg \gamma^{-1} \qquad \sigma_v^2 = \frac{D}{\gamma}$ saturation des fluctuations.

5) Thermalisation : $\langle E \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} kT$

~~et~~ $\langle \frac{1}{2} m v^2 \rangle = \frac{1}{2} m \frac{D}{\gamma}$ d'où $kT = \frac{mD}{\gamma}$

$$\Rightarrow \boxed{D = \frac{kT\gamma}{m}}$$ fluctuation - dissipation. 1

B- Bruit coloré

$$\frac{dy}{dt} = h(y) + \Gamma(t)$$

avec $\langle \Gamma(t) \rangle = 0$ et $\langle \Gamma(t)\Gamma(t') \rangle = \frac{D}{\gamma} e^{-\gamma|t_1 - t_2|}$.

1) $\langle \Gamma(t)\Gamma(t') \rangle$ n'est plus ponctuel, il y a donc une mémoire. Donc le processus n'est pas Markovien. 0,5

2) $$\frac{dy}{dt} = h(y) + \eta(t)$$ ~~le bruit Γ est blanc donc le processus est Markovien.~~

$$\frac{d\eta}{dt} = -\gamma \eta(t) + \Gamma(t)$$

~~Il suffit de montrer~~

a) On a des eq. du 1er ordre. Le bruit est blanc. Donc le processus est Markovien. ~~1~~ 0,5

b) Il suffit de montrer que les corrélations de $\eta(t)$ sont, dans la limite des grands temps, de la même forme que celles de $\hat{\Gamma}(t)$.

Car il est clair que : $\langle \eta(t)\eta(t')\rangle \underset{t\to\infty}{\simeq} \frac{D}{\gamma}\, e^{-\gamma|t-t'|}$.

1

II- Cinétique d'un processus de croissance.

1) 1

$$\begin{tikzcd} n-1 \arrow[r, bend right, "\beta(n-1)"'] & n \arrow[l, bend right, "\alpha n"'] \arrow[r, bend right, "\beta n"'] & n+1 \arrow[l, bend right, "\alpha(n+1)"'] \end{tikzcd}$$

2)

On a
$$\frac{dp_n(t)}{dt} = p_n(t)\left[-\alpha n - \beta n\right] + p_{n+1}(t)\left[\alpha(n+1)\right] + p_{n-1}(t)\left[\beta(n-1)\right].$$
1

3) On a
$$\sum_{n=0}^{\infty} n\,\frac{dp_n(t)}{dt} = \frac{d\langle n\rangle}{dt}$$

$$= \sum_{n=0}^{\infty} n\,(-\alpha n - \beta n)\,p_n(t) + \sum_{n=0}^{\infty} \alpha(n+1)\,p_{n+1}(t)\,n + \sum_{n=0}^{\infty} \beta(n-1)\,p_{n-1}(t)\,n.$$

$$\cancel{= -(\alpha+\beta)\langle n\rangle}$$

$$= \sum_{n=0}^{\infty}\left[-(\alpha+\beta)n^2 + \alpha\, n(n-1) + \beta\, n(n+1)\right] p_n(t).$$

$$= \sum_{n=0}^{\infty}\left[-\alpha n + \beta n\right] p_n(t) = (\beta-\alpha)\langle n\rangle.$$

$$\Rightarrow \boxed{\langle n(t)\rangle = n_0\, e^{(\beta-\alpha)t}}$$
2

4) $$\sum_{n=0}^{\infty} n^2 \frac{dp_n}{dt} = \sum_{n=0}^{\infty} n^2(-\alpha n-\beta n)\,p_n + \sum_{n=0}^{\infty} \alpha\, n^2(n+1)\,p_{n+1} + \sum_{n=0}^{\infty} \beta\, n^2(n-1)\,p_{n-1}$$

$$= \sum_{n=0}^{\infty} \left[-(\alpha+\beta)n^3 + \alpha(n-1)^2 n + \beta(n+1)^2 n\right] p_n(t)$$

$$= \sum_{n=0}^{\infty} \left[-(\alpha+\beta)n^3 + \alpha n(n^2-2n+1) + \beta(n^2+2n+1)n\right]$$

$$= \sum_{n=0}^{\infty} \left[\underline{-(\alpha+\beta)n^3} + \underline{\alpha n^3} - \underline{2\alpha n^2} + \alpha n + \underline{\beta n^3} + \underline{2\beta n^2} + \beta n\right]$$

$$= \sum_{n=0}^{\infty} \left[2(\beta-\alpha)n^2 + (\alpha+\beta)n\right] = 2(\beta-\alpha)\langle n^2\rangle + (\alpha+\beta)\langle n\rangle.$$

D'où $$\frac{d}{dt}\left[\langle n^2\rangle - \langle n\rangle^2\right] = 2(\beta-\alpha)\langle n^2\rangle + (\alpha+\beta)\langle n\rangle - 2\langle n\rangle \frac{d\langle n\rangle}{dt}$$

soit $$\boxed{\frac{d\sigma^2}{dt} = 2(\beta-\alpha)\,\sigma^2 + (\alpha+\beta)\langle n\rangle}$$ 3

(3)

5) a) $G(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$

$G(z=0,t) = p_0(t)$

(0,5)

b) Con a: $G(z,t) = \sum_{n=0}^{\infty} p_n(t) z^n$

$$\Rightarrow \frac{\partial G(z,t)}{\partial t} = \sum_{n=0}^{\infty} \frac{dp_n(t)}{dt} z^n$$

$$= \sum_{n=0}^{\infty} p_n(t)\left[-\alpha n - \beta n\right] z^n + \sum_{n=0}^{\infty} p_{n+1}(t)\, \alpha (n+1) z^n$$

$$+ \sum_{n=0}^{\infty} p_{n-1}(t)\, \beta (n-1) z^n$$

$$= -(\alpha+\beta) \sum_{n=0}^{\infty} p_n(t)\, z \frac{\partial}{\partial z} z^n + \sum_{n=0}^{\infty} p_{n+1}(t)\, \alpha \frac{\partial}{\partial z} z^{n+1}$$

$$+ \sum_{n=0}^{\infty} p_{n-1}(t)\, \beta z^{2} \frac{\partial}{\partial z} z^{(n-1)}$$

$$= -(\alpha+\beta) z \frac{\partial G}{\partial z} + \alpha \frac{\partial G}{\partial z} + \beta z^2 \frac{\partial G}{\partial z}.$$

$$= \left[-(\alpha+\beta) z + \alpha + \beta z^2\right] \frac{\partial G}{\partial z}.$$

(2)

$$\frac{\partial G}{\partial t} - \left[-(\alpha+\beta)z + \alpha + \beta z^2\right]\frac{\partial G}{\partial z} = 0$$

avec $G(z,0) = z^{n_0}$

$$\begin{cases} \dfrac{\partial z}{\partial s} = -\left[-(\alpha+\beta)z + \alpha + \beta z^2\right] \\ \dfrac{\partial t}{\partial s} = 1. \end{cases}$$

1

$$\Rightarrow \frac{\partial z}{\beta z^2 - (\alpha+\beta)z + \alpha.} = -1$$

on a : $z_{1,2} = \dfrac{(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{2\beta.} = \dfrac{(\alpha+\beta) \pm (\alpha-\beta)}{2\beta.}$

soit $z_1 = \dfrac{\alpha}{\beta}$ et $z_2 = 1$

$$\Rightarrow \frac{\partial z}{\beta(z-z_1)(z-z_2)} = -\partial s.$$

soit $\dfrac{\partial z}{\beta}\left[\dfrac{1}{z_1-z_2}\right]\left[\dfrac{1}{z-z_1} - \dfrac{1}{z-z_2}\right] = -\partial s$

avec $\dfrac{1}{\beta}\,\dfrac{1}{z_1-z_2} = \dfrac{1}{\beta}\,\dfrac{1}{\frac{\alpha}{\beta}-1} = \dfrac{1}{\alpha-\beta.}$

1

D'où $\begin{cases} \ln \dfrac{|z-z_1|}{|z-z_2|} = -(\alpha-\beta)s + K \\ t = s + c. \end{cases}$

$$\Rightarrow \frac{z-\frac{\alpha}{\beta}}{z-1} = e^{-(\alpha-\beta)t} k \quad \text{soit :}$$

$$\boxed{\frac{z-\frac{\alpha}{\beta}}{z-1} e^{(\alpha-\beta)t} = C}$$ 1

d). $$\frac{z(t=0)-\frac{\alpha}{\beta}}{z(t=1)-1} = \frac{z(t)-\frac{\alpha}{\beta}}{z(t)-1} e^{(\alpha-\beta)t}.$$

$$\Rightarrow z(t=0) = \frac{\frac{\alpha}{\beta} - \frac{z(t)-\frac{\alpha}{\beta}}{z(t)-1} e^{(\alpha-\beta)t}}{1 - \frac{z(t)-\frac{\alpha}{\beta}}{z(t)-1} e^{(\alpha-\beta)t}.}$$ 1

e) On a $\frac{dG(s)}{ds} = 0 \Rightarrow G(s) = G(0) = G(z(0), t=0)$

Avec $G(z, t=0) = z^{n_0}$

on a : $G(s) = G(z(t=0), t=0) = [z(t=0)]^{n_0}$ 1

b) $p_0(t) = G(z=0, t) = G(z(t=0)$ pour $z=0, t=0)$

or pour $z=0$ $\quad z(t=0) = \dfrac{\frac{\alpha}{\beta} - \frac{\left(-\frac{\alpha}{\beta}\right)}{-1} e^{(\alpha-\beta)t}}{1 - \frac{\left(-\frac{\alpha}{\beta}\right)}{-1} e^{(\alpha-\beta)t}}$

soit $z(t=0) = \dfrac{\frac{\alpha}{\beta} - \frac{\alpha}{\beta} e^{(\alpha-\beta)t}}{1 - \frac{\alpha}{\beta} e^{(\alpha-\beta)t}} = \dfrac{\alpha - \alpha e^{(\alpha-\beta)t}}{\beta - \alpha e^{(\alpha-\beta)t}}$.

et $p_0(t) = G(z(t=0).$ pour $z=0, t=0)$

$= \left(z(t=0) \text{ pour } z=0 \right)^{n_0}$

$= \left(\dfrac{\alpha - \alpha e^{(\alpha-\beta)t}}{\beta - \alpha e^{(\alpha-\beta)t}} \right)^{n_0}$ 2

rem $\alpha < \beta$. $\quad p_0(t) \to \left(\frac{\alpha}{\beta}\right)^{n_0}$ si $t \to +\infty$. ~~extinction possible~~

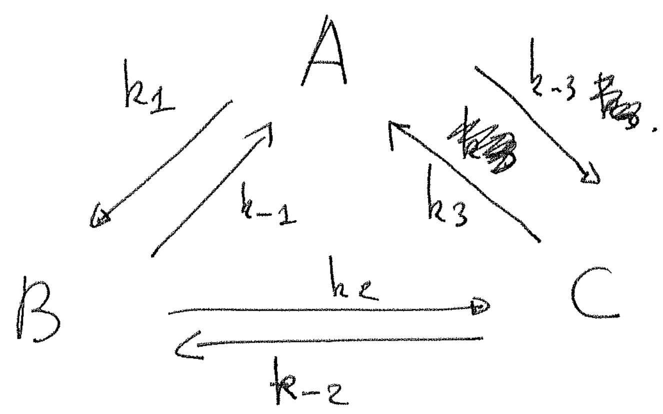
reste fini. 1

$\alpha > \beta$ $\quad p_0(t) \to 1$ si $t \to +\infty$.

extinction.

III – Modèle à 3 états d'une enzyme

1)



$$\frac{dp_A(t)}{dt} = k_3\, p_C + k_{-1}\, p_B - (k_1 + k_{-3})\, p_A$$

$$\frac{dp_B(t)}{dt} = k_1\, p_A + k_{-2}\, p_C - (k_{-1} + k_2)\, p_B$$ 1

$$\frac{dp_C(t)}{dt} = k_{-3}\, p_A + k_2\, p_B - (k_3 + k_{-2})\, p_C.$$

2).
$$\begin{aligned}
\frac{d\,pt}{dt} = \; & k_1 (n_A + 1)\, pt(n_A+1, n_B-1, n_C) + \\
& k_{-1} (n_B+1)\, pt(n_A-1, n_B+1, n_C) + \\
& k_2 (n_B+1)\, pt(n_A, n_B+1, n_C-1) + \\
& k_{-2} (n_C+1)\, pt(n_A, n_B-1, n_C+1) + \\
& k_3 (n_C+1)\, pt(n_A-1, n_B, n_C+1) + \\
& k_{-3} (n_A+1)\, pt(n_A+1, n_B, n_C-1) - \\
& [k_1 n_A + k_2 n_B + k_3 n_C + k_{-1} n_B + k_{-2} n_C + k_{-3} n_A]\, pt(n_A, n_B, n_C)
\end{aligned}$$ 2

3) $p_t(n_A, n_B, n_C) = \dfrac{N!}{\underbrace{n_A!\, n_B!\, n_C!}} \left[p_A(t)\right]^{n_A} \left[p_B(t)\right]^{n_B} \left[p_C(t)\right]^{n_C}$ 8

nb de fasons de choisir n_A molécules dans A 1

n_B ———— B

n_C ———— C.

4) $\langle n_i \rangle = N p_i \qquad i \in \{A, B, C\}$.

$\langle (n_i - \langle n_i \rangle)^2 \rangle = N p_i (1 - p_i)$ 2

$\langle (n_i - \langle n_i \rangle)(n_j - \langle n_j \rangle) \rangle = - N p_i p_j$